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Note on Elzaki Transform of Distributions and Certain Space of Boehmians

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NOTE ON ELZAKI TRANSFORM OF DISTRIBUTIONS AND CERTAIN SPACE OF BOEHMIANS

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Note on Elzaki Transform of Distributions and Certain Space of Boehmians

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Abstract - The Elzaki transform transform was discussed in [19] as a motivation of the classical Sumudu transform. In this article, we extend the Elzaki transform to a space of tempered distributions (distributions of slow growth) by known kernel method. Further, we establish two spaces of Boehmians so that the Elzaki transform is well defined . Certain theorems are established in some details.

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I. Introduction

In order to solve differential equations, several integral transforms were extensively used and applied in theory and application such as the Laplace, Fourier, Mellin, Hankel and Sumudu transforms, to name but a few. In the sequence of these transforms, rescently, Elzaki,T. and Elzaki, S. [17,18,19] introduced a motivation of the Sumudu transform [14 -16] and applied it to the solution of ordinary and partial differential equations as well.

The Elzaki transform over the set functions is defined by

$$A=\left\{f(t):\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{t/\tau_j},t\in(-1)^j\times(0,\infty)\right\} \quad (1)$$

by the formula

$$\tilde{f}(z)=Ef(z)=:\int^{\infty} zf(t)\,e^{\frac{-t}{z}}dt,z\in(-\tau_1,\tau_2). \quad (2)$$

The general properties of Elzaki transforms are found in above citations. In fact there is a relationship between Elzaki transform and some other transforms. In particular, the strong relationship between the Elzaki transform and Laplace transform was already proved in [19] which can be decribed as follows. Let f be a function of exponential order and Lf and Ef be the Laplace and Elzaki transforms of f, respectively, then

$$Ef(z)=zLf\left(\frac{1}{z}\right).$$

and hence

$$Lf\left(\frac{1}{z}\right)=zE\left(\frac{1}{z}\right).$$

The following are needful in the sequel.

(1) If a and b are non-negative real numbers then

$$E(af(t)+bg(t))(z)=aEf(z)+bEg(z).$$

(2) $\lim_{t\to0}f(t)=\lim_{z\to0}Ef(z)=f(0).$

II. Elzaki Transform of Boehmians

The minimal structure necessary for the construction of Boehmians consists of the following: (1) A nonempty set A ; (2) A commutative semigroup $(B,\star)$; (3) An operation $\star:A\times B\to A$ such that for each $x\in A$ and $s_1,s_2,\in B,x\star(s_1\star s_2)=(x\star s_1)\star s_2$; (3) A collection $\Delta\subset B^N$ such that: (a) If $x,y\in A,(s_n)\in\Delta,x\star s_n=y\star s_n$ for all then $x=y,$; (b) If $(s_n),(t_n)\in\Delta$, then $(s_n\star t_n)\in\Delta$.

Elements of Δ are called delta sequences. Consider

$$Q=\{(x_n,s_n):x_n\in A,(s_n)\in\Delta,x_n\star s_m=x_m\star s_n,\forall m,n\in\mathbf{N}\}.$$

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Global Journal of Science Frontier Research (F) Volume XII Issue I Version I January 2012

If $(x_n, s_n), (y_n, t_n) \in Q, x_n \star t_m = y_m \star s_n, \forall m, n \in \mathbf{N}$, then we say $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence relation in Q. The space of equivalence clases in Q is denoted by β. Elements of β are called Boehmians. Between A and β there is a canonical embedding expressed as $x \rightarrow \frac{x \star s_n}{s_n}$. The operation $\star$ can be extended to $\beta \times A$ by $\frac{x_n}{s_n} \star t = \frac{x_n \star t}{s_n}$. The relationship between the notion of convergence and the product $\star$ is given by:

(1) If $f_n \rightarrow f$ as $n \rightarrow \infty$ in A and, $\phi \in B$ is any fixed element, then $f_n \star \phi \rightarrow f \star \phi$ in A (as $n \rightarrow \infty$); (ii) If $f_n \rightarrow f$ as $n \rightarrow \infty$ in A and $(\delta_n) \in \Delta$, then $f_n \star \delta_n \rightarrow f$ in A (as $n \rightarrow \infty$).

The operation $\star$ is extended to $\beta \times B$ as follows: If $[\frac{f_n}{s_n}] \in \beta$ and $\phi \in B$,then $[\frac{f_n}{s_n}] \star \phi = [\frac{f_n \star \phi}{s_n}]$. Convergence in β is defined as

(1) : A sequence (h_n) in β is said to be δ convergent to h in β, $h_n \xrightarrow{\delta} h$, if there exists $(s_n) \in \Delta$ such that $(h_n \star s_n), (h \star s_n) \in A, \forall k, n \in \mathbf{N}$, and $(h_n \star s_k) \rightarrow (h \star s_k)$ as $n \rightarrow \infty$, in A, for every $k \in \mathbf{N}$.

(2) : A sequence (h_n) in β is said to be Δ convergent to h in β, $h_n \xrightarrow{\Delta} h$, if there exists a $(s_n) \in \Delta$ such that $(h_n - h) \star s_n \in A, \forall n \in \mathbf{N}$, and $(h_n - h) \star s_n \rightarrow 0$ as $n \rightarrow \infty$ in A. For further details, we refer to $[1-8, 10, 11, 13]$.

The convolution product between two functions u and v is given by the integral

$$(u * v)(y) = \int_0^\infty u(y - x) v(x) dx \qquad (3)$$

or, equivalently,

$$(u * v)(y) = \int u(t) \tau_y \tilde{v}(t) dt, \qquad (4)$$

where

$$\tilde{v}(t) = v(-t) \text{ and } \tau_y v(t) = v(t - y).$$

Lemma 2.1. $E(u * v)(z) = \frac{1}{z}(Eu)(z)(Ev)(z)$.

Proof See [19, Theo.2-6] :

Denote by S the space of all complex valued functions $s(t)$ that are infinitely smooth and are such that, as $|t| \rightarrow \infty$, they and their partial derivatives decrease to zero faster than every power of $\frac{1}{|t|}$.This required behaviour as $|t| \rightarrow \infty$ can also be stated in the following alternative way. For t one-dimensional, every function $s(t) \in S$ satisfies the infinite set of inequalities

$$\left| t^m s^{(k)}(t) \right| \leq C_{mk}, t \in (0, \infty), \qquad (5)$$

where m and k run through all non negative integers. The elements of S are called testing functions of rapid descents. S is a linear space. The dual space of S is denoted by $\acute{S}$. A distribution $u \in \acute{S}$ is said to be tempered distribution or distribution of slow growth.

Let $\mathbf{R}_+$ be the field of positive real numbers and z be arbitrary but fixed in $\mathbf{R}_+$ then

$$D_t^k \left(z e^{\frac{-t}{z}} \right) = (-1)^k z^{1-k} e^{\frac{-t}{z}}, k = 1, 2, ...$$

Hence for arbitrary but fixedz $z \in \mathbf{R}_+$, we get

$$\left| t^m D_t^k \left(z e^{\frac{-t}{z}} \right) \right| = \left| t^m z^{1-k} e^{\frac{-t}{z}} \right| < \infty, 0 < t < \infty. \qquad (6)$$

By aid of (6) we define the Elzaki transform of $u \in \acute{S}$ by kernel method as

$$Ef(z) = \left\langle u(t), z e^{\frac{-t}{z}} \right\rangle. \qquad (7)$$

$t, z \in \mathbf{R}_+$.

Global Journal of Science Frontier Research (F) Volume XII Issue I Version I January 2012

Denote by D the space of test functions of compact supports on $\mathbf{R}_+$ then

Definition 2.2 Let $u \in \acute{S}$ and $s \in D$ then we define the convolution $u * s$ to be C^∞ function such that

$$(u * v)(y) = \langle u, \tau_y \tilde{v} \rangle, \tag{8}$$

where $\tilde{v}(t) = v(-t)$ and $\tau_y v(t) = v(t-y), t \in R_+$. Equ(8) can also be written as

$$(u * v)(y) = \langle u(t), v(t-y) \rangle \tag{9}$$

Definition 2.3. The convolution of two tempered distributions $u, v \in \acute{S}$ is defined as an element in $\acute{S}$ through

$$\langle u * v, s \rangle = \langle u(y), \langle v(t), \phi(t+y) \rangle \rangle, s \in D. \tag{10}$$

It can be noted that if $u \in \acute{S}, v \in S$ then $u * v \in O_m$, where O_m is the space of multipliers for $\acute{S}$. In fact $O_m \subset \acute{S}$. This, establishes the following lemma.

Lemma 2.4. If $u \in \acute{S}, s \in D$ then $u * s \in \acute{S}$.

Lemma 2.5. If $u \in \acute{S}, s_1, s_2 \in D$ then

$$(u * s_1) * s_2 = u * (s_1 * s_2).$$

Proof. Since $D \subset S, u * s_1 \in C^\infty$ and hence $(u * s_1) * s_2 \in C^\infty, u * (s_1 * s_2) \in C^\infty$. Aso, $u \in \acute{S}, s_1 \in D \subset S \subset \acute{S}$, implies

$$u * s_1 \in \acute{S}.$$

We write

$$\begin{aligned} ((u * s_1) * s_2)(y) &= \langle u * s_1, \tau_y \tilde{s}_2 \rangle \\ &= \langle u(t), \langle s_1(x), \tau_y \tilde{s}_2(t+x) \rangle \rangle \\ &= \langle u(t), (s_1 * s_2)(y-t) \rangle \\ &= (u * (s_1 * s_2))(y). \end{aligned}$$

Hence

$$(u * s_1) * s_2 = u * (s_1 * s_2).$$

This completes the proof.

Lemma 2.6 If $u_1, u_2 \in \acute{S}, s \in D$ and $\alpha \in R$ then we have $(1)\,(u_1 + u_2) * s = u_1 * s + u_2 * s.$; $(2)\,\alpha(u_1 * s) = (\alpha u_1) * s = u_1 * (\alpha s)$. Let Δ be the collection of all sequences (r_n) from D such that Equ. $(11-13)$ satisfies.

$$\int_{\mathbf{R}_+} r_n(t)\, dt = 1 \tag{11}$$

$$\int_{\mathbf{R}_+} |r_n(t)|\, dt < M, M \in \mathbf{R}_+ \tag{12}$$

$$\operatorname{supp} r_n(t) \to 0 \text{ as } n \to \infty \tag{13}$$

Sequences from Δ are called delta sequences.

Lemma 2.7. If $u_n \to u \ is\ S\ as\ n \to \infty$ then

$$u_n * s \to u * s \text{ as } n \to \infty \text{ in } \acute{S}, s \in D.$$

Lemma 2.8. If $u_n \to u\ in\ \acute{S}$ as $n \to \infty\ then\ u_n * r_n \to u$ as $n \to \infty$ for each $(r_n) \in \Delta$.

The described Boehmian space is denoted by $O\left(\acute{S}, D, \Delta\right)$. Next, we describe another Boehmian space as follows.

Let H be the set of all Elzaki transforms of tempered distributions from $\acute{S}$. That is, for each $h \in H$, there is $u \in \acute{S}$ such that $h = Eu$. Moreover, $h_n \to h$ in H if there are $u_n, u \in \acute{S}$ such that $u_n \to u$ in $\acute{S}$.

Define a mapping $\bullet$ between $h \in H$ and $s \in D$ by

$$(h \bullet s)(z) = h(z) \int e^{\frac{-t}{z}} s(t)\, dt \tag{14}$$

Lemma 2.9. Let $h \in H$ such that $h = Eu, u \in \acute{S}$ and $s \in D$ then

$$E(u * s)(z) = (h \bullet s)(z).$$

Global Journal of Science Frontier Research (F) Volume XII Issue I Version I January 2012

Proof. Using definitions and Leibnitz' rule and change of varibles yields

$$\begin{aligned} E(u*s)(z) &= \int s(t)\,dt \int u(y)\, z e^{-\frac{t+y}{z}}\,dy \\ &= \int u(y)\, z e^{\frac{-y}{z}}\,dy \int e^{\frac{-t}{z}} s(t)\,dt \\ &= h(z) \int e^{\frac{-t}{z}} s(t)\,dt \\ &= (h \bullet s)(z). \end{aligned}$$

Hence the Lemma.

Following lemmas are straightforward. We avoid same details.

Lemma 2.10 If $h \in H, s \in D$ then $h \bullet s \in H$.

Note that if $h \in H$ then $h = Eu$, for some $u \in \acute{S}$. Therefore $h \bullet s = Eu \bullet s = E(u*s)$, by Lemma 2.9. Since $u * s \in \acute{S}$, the lemma follows.

Lemma 2.11. If $h \in H, s \in D$ then $E^{-1}(h \bullet s) = E^{-1} h * \phi$ where E^{-1} is the inverse Elzaki transform

Proof. Let $u \in \acute{S}$ such that $Eu = h$ then

$$E(u*s) = h \bullet s.$$

Hence, employing E^{-1} on both sides yields $E^{-1}(h \bullet s) = u * s = E^{-1} h * s$.

Lemma 2.12. If $h_1, h_2 \in H, s_1, s_2 \in$ then

$$(h_1 + h_2) \bullet s = h_1 \bullet s + h_2 \bullet s; \quad (2)\, h \bullet (s_1 * s_2) = (h \bullet s_1) \bullet s_2.$$

Lemma 2.13. If $h_n \to h$ and $s \in D$ then $h_n \bullet s \to h \bullet s$.

Lemma 2.14. If $h_n \to h$ in H and $(r_n) \in \Delta$ then

$$h_n \bullet r_n \to h \text{ as } n \to \infty.$$

The space $O(H, D, \Delta)$ can therefore be regarded as a Boehmian space.

III. Elzaki Transform of Boehmians

Let $\beta_1 = \left[\frac{u_n}{s_n}\right] \in O\left(\acute{S}, D, \Delta\right)$ then we define the extended Elzaki transform of β_1 as

$$\hat{E}\left[\frac{u_n}{r_n}\right] = \left[\frac{Eu_n}{r_n}\right] \in O(H, D, \Delta), \tag{15}$$

where $(r_n) \in \Delta$.

Theorem 3.1. $\hat{E} : O\left(\acute{S}, D, \Delta\right) \to O(H, D, \Delta)$ is well defined.

Proof: Let $\left[\frac{u_n}{r_n}\right] = \left[\frac{v_n}{\psi_n}\right]$ in $O\left(\acute{S}, D, \Delta\right)$ then

$$u_n * \psi_m = v_m * r_n = v_n * r_m.$$

Employing E on both sides,

$$(Eu_n)(z) \int \psi_m(t)\, e^{\frac{-t}{z}}\,dt = (Ev_n)(z) \int r_m(t)\, e^{\frac{-t}{z}}\,dt.$$

Hence,

$$Eu_n \bullet \psi_m = Ev_n \bullet r_m.$$

That is,

$$\frac{Eu_n}{r_n} \sim \frac{Ev_n}{\psi_n}.$$

Therefore,

$$\left[\frac{Eu_n}{r_n}\right] = \left[\frac{Ev_n}{\psi_n}\right].$$

This completes the proof of the theorem.

Theorem 3.2. $\hat{E} : O\left(\acute{S}, D, \Delta\right) \to O\left(H, D, \Delta\right)$ is linear

Proof. Let $\left[\frac{u_n}{r_n}\right], \left[\frac{v_n}{\psi_n}\right]$. From definitions and Equ. (15) we get

$$\begin{aligned}\hat{E}\left(\left[\frac{u_n}{r_n}\right]+\left[\frac{v_n}{\psi_n}\right]\right) &= \hat{E}\left(\left[\frac{u_n*\psi_n+v_n*r_n}{r_n*\psi_n}\right]\right)\\ &= \left[\frac{E\left(u_n*\psi_n+v_n*r_n\right)}{r_n*\psi_n}\right]\\ &= \left[\frac{E\left(u_n*\psi_n\right)+E\left(v_n*r_n\right)}{r_n*\psi_n}\right]\\ &= \left[\frac{Eu_n\bullet\psi_n+Ev_n\bullet r_n}{r_n*\psi_n}\right]\\ &= \left[\frac{Eu_n}{r_n}\right]+\left[\frac{Ev_n}{\psi_n}\right].\end{aligned}$$

Hence

$$\hat{E}\left(\left[\frac{u_n}{r_n}\right]+\left[\frac{v_n}{\psi_n}\right]\right)=\hat{E}\left[\frac{u_n}{r_n}\right]+\hat{E}\left[\frac{v_n}{\psi_n}\right].$$

Also, if $\alpha \in \mathbf{R}_+$ then

$$\alpha\hat{E}\left[\frac{u_n}{r_n}\right]=\alpha\left[\frac{Eu_n}{r_n}\right]=\left[\frac{E\left(\alpha u_n\right)}{r_n}\right].$$

Hence

$$\alpha\hat{E}\left[\frac{u_n}{r_n}\right]=\hat{E}\left(\alpha\left[\frac{u_n}{r_n}\right]\right).$$

This completes the proof.

Theorem 3.3. $\hat{E}$ is one-one.

Proof. Let $\beta_1, \beta_2 \in O\left(\acute{S}, D, \Delta\right)$ such $\beta_1 = \left[\frac{u_n}{r_n}\right]$ and $\beta_2 = \left[\frac{v_n}{\psi_n}\right]$.

Assume $E\beta_1 = E\beta_2$ then $\left[\frac{Eu_n}{r_n}\right] = \left[\frac{Ev_n}{\psi_n}\right]$. That is,

$$Eu_n\bullet\psi_n = Ev_m\bullet r_n.$$

Using Lemma 2.9,

$$E\left(u_n*\psi_m\right)=E\left(v_m*r_n\right).$$

Therefore

$$u_n*\psi_m=v_m*r_n.$$

Hence

$$\frac{u_n}{r_n}\sim\frac{v_n}{\psi_n}$$

and

$$\left[\frac{u_n}{r_n}\right]\sim\left[\frac{v_n}{\psi_n}\right].$$

This completes the proof of the lemma.

Theorem 3.4. $\hat{E} : O\left(\acute{S}, D, \Delta\right) \to O\left(H, D, \Delta\right)$ is onto.

Proof. Let $\left[\frac{h_n}{r_n}\right] \in O\left(H, D, \Delta\right)$ then

$$h_n = Eu_n,$$

Global Journal of Science Frontier Research (F) Volume XII Issue I Version I January 2012

for all n. $\left[\frac{u_n}{r_n}\right]$ is in $O\left(\acute{S}, D, \Delta\right)$ such that

$$\hat{E}\left[\frac{u_n}{r_n}\right] = \left[\frac{Eu_n}{r_n}\right] = \left[\frac{h_n}{r_n}\right].$$

Hence the theorem. Now, we de.ne the inverse $\hat{E}^{-1}$ by the relation

$$\hat{E}^{-1}\left[\frac{h_n}{r_n}\right] = \left[\frac{\hat{E}^{-1}h_n}{r_n}\right], \qquad (16)$$

for every $h_n \in O(H, D, \Delta)$.

Theorem 3.5. $\hat{E}^{-1} : O(H, D, \Delta) \to O\left(\acute{S}, D, \Delta\right)$ is well defined.

Theorem 3.6. $\hat{E}^{-1} : O(H, D, \Delta) \to O\left(\acute{S}, D, \Delta\right)$ is linear.

Theorem 3.7. $\hat{E}^{-1} : O(H, D, \Delta) \to O\left(\acute{S}, D, \Delta\right)$ is an isomorphism.

Proof of Theorem 3.5, 3.6, 3.7, are analogous to that of Theorem 3.1, 3.2, 3.3, and 3.4. Detailed proofs are avoided.

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